

Quantization and spacetime topology

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Abstract

We consider classical and quantum dynamics of a free particle in de Sitter's space-times with different topologies to see what happens to space-time singularities of removable type in quantum theory. We find analytic solution of the classical dynamics. The quantum dynamics is solved by finding an essentially self-adjoint representation of the algebra of observables integrable to the unitary representations of the symmetry group of each considered gravitational system. The dynamics of a massless particle is obtained in the zero-mass limit of the massive case. Our results indicate that taking account of global properties of space-time enables quantization of particle dynamics in all considered cases.

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I. INTRODUCTION

Cosmological data indicate that the Universe expands, so in the past it could be in a special state characterized by physical fields with extremally high densities. There are also theoretical indications that our universe emerged from a very special state: the well known class of solutions to the Einstein equations called the FLRW universes suggest that in the past our universe could be in a state with blowing up Riemann tensor components or scalar curvature and with blowing up energy density. There is a common believe that to analyse this state properly one should include quantum effects. The struggle for quantum gravity lasts about 70 years. There is a real progress, but we believe that one should first understand the nature of space-time singularities in a quantum context. This understanding may mean changing some of the principles underlying quantum mechanics or general relativity. The insight into the problem may be achieved by studying some suitable toy models which include both space-time singularities and quantum rules. In what follows we present results concerning one of such models which is quantization of dynamics of a test particle in singular and corresponding regular space-times.

Recently it was found [1, 2, 3, 4] that classical and quantum dynamics of a particle in a curved space-time seems to be sensitive to the topology of space-time. Our aim is examination of this dependence in details. It enables the understanding of the nature of removable type singularities of space-time.

We use the group theory oriented quantization (GTQ) scheme, which we have already applied to simple gravitational systems [1, 2, 3, 4, 5]. Our method is similar to the GTQ method initiated by Isham [6] and Kirillov [7].

In what follows we examine classical and quantum dynamics of a particle in two-dimensional space-times with different topologies. We carry out all calculations rigorously which enables complete discussion of considered problems. In the last section we make the argument that our results can be extended to higher dimensions.

In Sec. II we present the dynamics of a particle in regular space-time. Application of the standard GTQ method leads to well defined results.

Analyses of particle dynamics in singular space-time is carried out in Sec. III. The GTQ method needs some modification to be applicable in this case, since the relation between local and global properties of considered system cannot be directly modeled by mathematics connecting Lie group and its Lie algebra (consequently also at the level of representations). However, redefinition (for the purpose of quantization) of the notion of local symmetries of a gravitational system enables the quantization. The problem of quantization in this case is directly connected with the problem of space-time singularities. We present the solution in case space-time has singularities of removable type.

Secs. II and III deal with a particle with a non-zero mass.

In Sec. IV we present the dynamics of a massless particle. It is obtained in the zero-mass limit from the massive particle dynamics.

We conclude in Sec. V. The last section comprises the list of references.

II. PARTICLE ON HYPERBOLOID

The considered space-times, V_p and V_h , are of de Sitter's type. They are defined to be [5]

$$V_p = (\mathbb{R} \times \mathbb{R}, \hat{g}) \quad \text{and} \quad V_h = (\mathbb{R} \times \mathbb{S}, \hat{g}). \quad (2.1)$$

In both cases the metric $g_{\mu\nu} := (\hat{g})_{\mu\nu}$ ($\mu, \nu = 0, 1$) is defined by the line-element

$$ds^2 = dt^2 - \exp(2t/r) dx^2, \quad (2.2)$$

where r is a positive real constant.

It is clear that (2.1) includes all possible topologies of de Sitter's type space-times in two dimensions which makes our examination complete. V_p is a plane with global $(t, x) \in \mathbb{R}^2$ coordinates. V_h is defined to be a one-sheet hyperboloid embedded in 3d Minkowski space. There exists an isometric immersion map [8] of V_p into V_h

$$V_p \ni (t, x) \longrightarrow (y^0, y^1, y^2) \in V_h, \quad (2.3)$$

where

$$y^0 := r \sinh(t/r) + \frac{x^2}{2r} \exp(t/r), \quad y^1 := -r \cosh(t/r) + \frac{x^2}{2r} \exp(t/r), \quad y^2 := -x \exp(t/r),$$

and where

$$(y^2)^2 + (y^1)^2 - (y^0)^2 = r^2. \quad (2.4)$$

Eq. (2.3) defines a map of V_p onto a simply connected non-compact half of V_h . Thus, V_p is just a part of V_h . One can check that the induced metric on V_h coincides with the metric defined by (2.2).

It is known [8] that V_p is geodesically incomplete. However, all incomplete geodesics in V_p can be extended to complete ones in V_h , i.e. V_p has removable type singularities. V_p and V_h are the simplest examples of space-times with constant curvatures and with noncompact and compact spaces, respectively.

An action integral, \mathcal{A} , describing a free relativistic particle of mass m in gravitational field $g_{\mu\nu}$ is proportional to the length of a particle world-line and is given by

$$\mathcal{A} = \int_{\tau_1}^{\tau_2} L(\tau) d\tau, \quad L(\tau) := -m \sqrt{g_{\mu\nu}(x^0(\tau), x^1(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau)}, \quad (2.5)$$

where τ is an evolution parameter, x^μ are space-time coordinates and $\dot{x}^\mu := dx^\mu/d\tau$. It is assumed that $\dot{x}^0 > 0$, i.e., x^0 has interpretation of time monotonically increasing with τ .

The Lagrangian (2.5) is invariant under the reparametrization $\tau \rightarrow f(\tau)$. This gauge symmetry leads to the constraint

$$G := g^{\mu\nu} p_\mu p_\nu - m^2 = 0, \quad (2.6)$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and $p_\mu := \partial L / \partial \dot{x}^\mu$ are canonical momenta.

Since we assume that a free particle does not modify the geometry of space-time, the local symmetry of the system is defined by the set of all Killing vector fields of space-time (which is also the local symmetry of the Lagrangian L). The corresponding dynamical integrals have the form [9]

$$D = p_\mu X^\mu, \quad \mu = 0, 1, \quad (2.7)$$

where X^μ is a Killing vector field.

The physical phase-space Γ is defined to be the space of all particle trajectories [10] consistent with the dynamics of a particle and with the constraint (2.6).

A. Classical dynamics

Since we consider the dynamics of a material particle (i.e. moving along timelike geodesics) on the hyperboloid (2.4), the symmetry group of V_h system is the proper orthochronous Lorentz group $SO_0(1, 2)$.

If we parametrize (2.4) as follows

$$y^0 = -\frac{r \cos \rho/r}{\sin \rho/r}, \quad y^1 = \frac{r \cos \theta/r}{\sin \rho/r}, \quad y^2 = \frac{r \sin \theta/r}{\sin \rho/r}, \quad (2.8)$$

where $0 < \rho < \pi r$, $0 \leq \theta < 2\pi r$, the line-element on the hyperboloid (2.4) reads

$$ds^2 = (d\rho^2 - d\theta^2) \sin^{-2}(\rho/r), \quad (2.9)$$

and the Lagrangian (2.5) is given by

$$L = -m \sqrt{\frac{\dot{\rho}^2 - \dot{\theta}^2}{\sin^2(\rho/r)}}. \quad (2.10)$$

Since we consider only timelike trajectories ($|\dot{\rho}| > |\dot{\theta}|$), the Lagrangian (2.10) is well defined.

The infinitesimal transformations of $SO_0(1, 2)$ group (rotation and two boosts) have the form

$$\begin{aligned} (\rho, \theta) &\longrightarrow (\rho, \theta + a_0 r), \\ (\rho, \theta) &\longrightarrow (\rho - a_1 r \sin \rho/r \sin \theta/r, \theta + a_1 r \cos \rho/r \cos \theta/r), \\ (\rho, \theta) &\longrightarrow (\rho + a_2 r \sin \rho/r \cos \theta/r, \theta + a_2 r \cos \rho/r \sin \theta/r), \end{aligned} \quad (2.11)$$

where $(a_0, a_1, a_2) \in \mathbb{R}^3$ are small parameters.

The corresponding dynamical integrals (2.7) are

$$\begin{aligned} J_0 &= p_\theta r, \quad J_1 = -p_\rho r \sin \rho/r \sin \theta/r + p_\theta r \cos \rho/r \cos \theta/r, \\ J_2 &= p_\rho r \sin \rho/r \cos \theta/r + p_\theta r \cos \rho/r \sin \theta/r, \end{aligned} \quad (2.12)$$

where $p_\theta := \partial L / \partial \dot{\theta}$, $p_\rho := \partial L / \partial \dot{\rho}$ are canonical momenta.

One can check that the dynamical integrals (2.12) satisfy the commutation relations of $sl(2, \mathbb{R})$ algebra

$$\{J_0, J_1\} = -J_2, \quad \{J_0, J_2\} = J_1, \quad \{J_1, J_2\} = J_0. \quad (2.13)$$

The constraint (2.6) reads

$$(p_\rho^2 - p_\theta^2) \sin^2(\rho/r) = m^2. \quad (2.14)$$

Making use of (2.12) we find that (2.14) relates the dynamical integrals

$$J_1^2 + J_2^2 - J_0^2 = \kappa^2, \quad \kappa := mr. \quad (2.15)$$

Eqs. (2.8) and (2.12) lead to equations for a particle trajectory

$$J_a y^a = 0, \quad J_2 y^1 - J_1 y^2 = r^2 p_\rho, \quad (2.16)$$

where $p_\rho < 0$, since we consider timelike trajectories.

Each point (J_0, J_1, J_2) of (2.15) defines uniquely a particle trajectory (2.16) on (2.4) admissible by the dynamics and consistent with the constraint (2.14). Thus, the one-sheet hyperboloid (2.15) defines the physical phase-space Γ_h . Since we intend to quantize the dynamics of a particle canonically, we should identify the symmetry group of Γ_h . At the level of space-time the symmetry group is $SO_0(1, 2)$. The phase-space (2.15) and space-time (2.4) have the same manifold structure, but it does not mean that $SO_0(1, 2)$ is the only possible group action on Γ_h . Since the phase-space (2.15) is not simply connected, we can choose any group with $sl(2, \mathbb{R})$ as its Lie algebra to be the symmetry group, i.e. any covering group of $SO_0(1, 2)$. This ambiguity is one of the sources of nonuniqueness of quantum dynamics considered in Sec. IIC.

B. Observables

We define classical observables to be smooth functions on phase-space satisfying the following conditions: (i) algebra of observables corresponds to the local symmetry of phase-space, (ii) observables specify particle trajectories admissible by the dynamics (V_p and V_h are integrable systems), and (iii) observables are gauge invariant, i.e. have vanishing Poisson's brackets with the constraint G , Eq. (2.6).

In what follows we do not carry out the Hamiltonian reduction explicitly. We make use of our Hamiltonian reduction scheme to gauge invariant variables presented in [2].

The canonical coordinates on phase-space are chosen in such a way that the classical observables are first order polynomials in one of the canonical coordinates. Such a choice enables, in the quantization procedure, solution of the operator-ordering problem by symmetrization. It also simplifies discussion of self-adjointness of quantum operators which in the linear case reduces to the solution of the first order linear differential equation (see, App. A).

C. Quantum dynamics

In case the global symmetry of a classical system is defined by a Lie group with its Lie algebra being isomorphic to the Lie algebra of a local symmetry of the system, application of the GTQ method is straightforward. It consists in finding an irreducible unitary representation of the symmetry group on a Hilbert space. The representation space provides the quantum states space. The application of Stone's theorem [11, 12] to the representation of one-parameter subgroups of the symmetry group leads to self-adjoint operators representing quantum observables. Alternatively, by quantization we mean finding an essentially self-adjoint representation of the algebra of observables (corresponding to the local symmetry of the system) on a dense subspace of a Hilbert space, integrable to the irreducible unitary representation of the symmetry group of the gravitational system.

Since our V_h system satisfies the above symmetry relationship, it can be quantized by making use of our GTQ method.

We choose J_0, J_1 and J_2 as the classical observables. One can easily verify that the criteria (i), (ii) and (iii) of Sec. IIB are satisfied. We parametrize the hyperboloid (2.15) as follows

$$J_0 = J, \quad J_1 = J \cos \beta - \kappa \sin \beta, \quad J_2 = J \sin \beta + \kappa \cos \beta, \quad (2.17)$$

where $J \in \mathbb{R}$ and $0 \leq \beta < 2\pi$.

In this new parametrization the observables are linear in the coordinate J .

One can check that the canonical commutation relation $\{J, \beta\} = 1$ leads to Eq. (2.13).

Making use of the Schrödinger representation for the canonical coordinates J and β (we set $\hbar = 1$ through the paper)

$$\beta \rightarrow \hat{\beta}\psi(\beta) := \beta\psi(\beta), \quad J \rightarrow \hat{J}\psi(\beta) := -i\frac{d}{d\beta}\psi(\beta),$$

and applying the symmetrization prescription to the products in (2.17)

$$J \cos \beta \rightarrow \frac{1}{2}(\hat{J} \cos \hat{\beta} + \cos \hat{\beta} \hat{J}), \quad J \sin \beta \rightarrow \frac{1}{2}(\hat{J} \sin \hat{\beta} + \sin \hat{\beta} \hat{J})$$

leads to

$$\hat{J}_0\psi(\beta) = \left[-i\frac{d}{d\beta}\right]\psi(\beta), \quad (2.18)$$

$$\hat{J}_1\psi(\beta) = \left[\cos \beta \hat{J}_0 - \left(\kappa - \frac{i}{2}\right) \sin \beta\right]\psi(\beta), \quad (2.19)$$

$$\hat{J}_2\psi(\beta) = \left[\sin \beta \hat{J}_0 + \left(\kappa - \frac{i}{2}\right) \cos \beta\right]\psi(\beta), \quad (2.20)$$

where $\psi \in \Omega_\theta \subset L^2(\mathbb{S})$, $\theta \in \mathbb{R}$, and where $L^2(\mathbb{S})$ is the space of square-integrable complex functions on a unit circle \mathbb{S} with the scalar product

$$\langle \psi_1 | \psi_2 \rangle := \int_0^{2\pi} d\beta \overline{\psi_1(\beta)} \psi_2(\beta). \quad (2.21)$$

The subspace Ω_θ is defined to be

$$\Omega_\theta := \{\psi \in L^2(\mathbb{S}) \mid \psi \in C^\infty[0, 2\pi], \psi^{(n)}(0) = e^{i\theta} \psi^{(n)}(2\pi), n = 0, 1, 2, \dots\}. \quad (2.22)$$

The representation (2.18 - 2.22) is parametrized by $\theta \in \mathbb{R}$.

The unbounded operators \hat{J}_a ($a = 0, 1, 2$) are well defined because Ω_θ is a dense subspace of the Hilbert space $L^2(\mathbb{S})$.

It is clear that Ω_θ is a common invariant domain for all \hat{J}_a and their products. One can verify that

$$[\hat{J}_a, \hat{J}_b]\psi = -i\widehat{\{J_a, J_b\}}\psi, \quad \psi \in \Omega_\theta, \quad (2.23)$$

and that the representation (2.18 - 2.22) is symmetric on Ω_θ . We prove in App. A that this representation is essentially self-adjoint.

It is known [2, 14, 15] that the parameter θ labels unitarily non-equivalent representations of $sl(2, \mathbb{R})$ algebra corresponding to the unitary representations of various covering groups of $SO_0(1, 2)$, so it is connected with the ambiguity in the choice of the symmetry group of phase-space considered at the end of Sec. IIA. For the purpose of clarity of discussion of the new ambiguity problem connected directly with singularities of space-time of V_p system, we carry out further discussion for a fixed value of θ . In what follows we put $\theta = 0$ which corresponds to the choice of $SO_0(1, 2)$. The ambiguities in quantization connected with the choice of θ and with the choice of a symmetry group of phase-space in general will be considered elsewhere [16].

The problem of finding representations of the group $SO_0(1, 2)$ was considered in 1947 by Bargmann [13] in the context of representation of $SU(1, 1)$ group. There exists two-to-one homomorphism of $SU(1, 1)$ group onto $SO_0(1, 2)$ with the kernel $\mathbb{Z}_2 := \{e, -e\}$, where e is the identity element of $SU(1, 1)$. Thus, the factor group $SU(1, 1)/\mathbb{Z}_2$ is isomorphic to $SO_0(1, 2)$.

Bargmann has constructed and classified all irreducible unitary representations of $SU(1, 1)$ group by making use of the multiplier representation method [13, 17]. These representations fall basically into three classes [13, 18]: principal series, complementary series and discrete series. Bargmann's classification is based on: (i) his special decomposition of $SU(1, 1)$ group (see, Eq. (4.12) of [13]) into a product of one-parameter subgroups one of which is a compact Abelian group with unitary representation having complete system of vectors and integral (corresponding to $SO_0(1, 2)$ group) or half-integral proper values, and (ii) his classification of irreducible representations of $su(1, 1)$ algebra.

To compare our representation within Bargmann's, we choose as a basis in $L^2(\mathbb{S})$ his basis

$$\phi_m(\beta) = (2\pi)^{-1/2} \exp(im\beta), \quad 0 \leq \beta < 2\pi, \quad m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}. \quad (2.24)$$

Since the algebras $so(1, 2)$, $su(1, 1)$ and $sl(2, \mathbb{R})$ are isomorphic [17], we make the comparison with Bargmann's representation at the level of algebra. Correspondingly, we examine the action of the operators \hat{J}_a and \hat{C} on the subspace $\Omega := \Omega_{\theta=0}$ spanned by the set of vectors (2.24). The operator \hat{C} corresponds to the Casimir operator C of $sl(2, \mathbb{R})$ algebra. C is defined to be [19]

$$C = J_1^2 + J_2^2 - J_0^2. \quad (2.25)$$

In our representation the operator \hat{C} reads

$$\hat{C}\psi = [\hat{J}_1^2 + \hat{J}_2^2 - \hat{J}_0^2]\psi = (\kappa^2 + 1/4)\psi, \quad \psi \in \Omega, \quad (2.26)$$

where the third term in (2.26) was obtained by making use of explicit formulas for \hat{J}_a , Eqs. (2.18 - 2.20).

It is easy to verify that the action of the operators \hat{J}_a on ϕ_m reads

$$\hat{J}_0\phi_m = m\phi_m, \quad m \in \mathbb{Z} \quad (2.27)$$

$$\hat{J}_1\phi_m = \frac{1}{2}(m + 1/2 + i\kappa)\phi_{m+1} + \frac{1}{2}(m - 1/2 - i\kappa)\phi_{m-1}, \quad (2.28)$$

$$\hat{J}_2\phi_m = -\frac{i}{2}(m + 1/2 + i\kappa)\phi_{m+1} + \frac{i}{2}(m - 1/2 - i\kappa)\phi_{m-1}. \quad (2.29)$$

At this stage we are ready to discuss the connection of our representation with Bargmann's. Direct comparison of Eqs. (2.26 - 2.29) with Bargmann's (6.14), (6.21) and (6.22) of Ref. [13] shows that the following identification is possible:

$$\hat{C} \equiv Q, \quad \kappa^2 \equiv q - 1/4, \quad \hat{J}_0 \equiv H_0 = i\Lambda_0, \quad \hat{J}_1 \equiv -H_1 = -i\Lambda_1, \quad \hat{J}_2 \equiv -H_2 = -i\Lambda_2, \quad (2.30)$$

where Q, q, H_a, Λ_a ($a = 0, 1, 2$) are Bargman's quantities used to define his representation of $su(1, 1)$ algebra.

The range of our parameter $\kappa = mr$ is $0 < \kappa < \infty$, so it corresponds to Bargmann's $1/4 < q < \infty$. Therefore, our representation is almost everywhere identical with Bargmann's

continuous class integral case (corresponding to $SO_0(1, 2)$ group) called C_q^0 with $1/4 \leq q < \infty$, which is also called the principal series of irreducible unitary representation of $SU(1, 1)$ group [18]. The only difference is that for massive particle $m > 0$, thus $\kappa = mr > 0$, so $q > 1/4$. The precise identity may occur, if taking the limit $\kappa \rightarrow 0$ can be given physical and mathematical sense in our formalism. We discuss this issue in Sec. IVA.

III. PARTICLE ON PLANE

A. Restrictions for classical dynamics

The Lagrangian (2.5) with the metric tensor defined by (2.2) reads

$$L = -m\sqrt{\dot{t}^2 - \dot{x}^2 \exp(2t/r)}, \quad (3.1)$$

where $t := x^0$, $x := x^1$, $\dot{t} = dt/d\tau$ and $\dot{x} = dx/d\tau$.

The local symmetries of L (and the infinitesimal transformations of V_p space-time) are defined by translations

$$(t, x) \longrightarrow (t, x + b_0), \quad (3.2)$$

space dilatations with time translations

$$(t, x) \longrightarrow (t - rb_1, x + xb_1), \quad (3.3)$$

and by the transformations

$$(t, x) \longrightarrow (t - 2rxb_2, x + (x^2 + r^2e^{-2t/r})b_2), \quad (3.4)$$

where $(b_0, b_1, b_2) \in \mathbb{R}^3$ are small parameters.

The Killing vector fields corresponding to the transformations (3.2 - 3.4) define, respectively, the dynamical integrals (2.7)

$$P = p_x, \quad K = -rp_t + xp_x, \quad M = -2rxp_t + (x^2 + r^2e^{-2t/r})p_x, \quad (3.5)$$

where $p_x = \partial L / \partial \dot{x}$, $p_t = \partial L / \partial \dot{t}$.

One can verify that the dynamical integrals (3.5) satisfy the commutation relations of $sl(2, \mathbb{R})$ algebra in the form

$$\{P, K\} = P, \quad \{K, M\} = M, \quad \{P, M\} = 2K. \quad (3.6)$$

The mass-shell condition (2.6) takes the form

$$p_t^2 - e^{-2t/r}p_x^2 = m^2, \quad (3.7)$$

which, due to (3.5), relates the dynamical integrals

$$K^2 - PM = \kappa^2, \quad \kappa = mr. \quad (3.8)$$

By analogy to V_h case one may expect that each triple (P, K, M) satisfying (3.8) determines a trajectory of a particle. However, not all such trajectories are consistent with particle dynamics:

For $P = 0$ there are two lines $K = \pm\kappa$ on the hyperboloid (3.8). Since by assumption $\dot{t} > 0$, we have $p_t = \partial L / \partial \dot{t} = -m\dot{t} (\dot{t} - \dot{x} \exp(2t/r))^{-1/2} < 0$. According to (3.5) $K - xP = -rp_t$, thus $K - xP > 0$, i.e. $K > 0$ for $P = 0$. Therefore, the line ($P = 0$, $K = -\kappa$) is not available for the dynamics. The hyperboloid (3.8) without this line defines the physical phase-space Γ_p , which is topologically equivalent to \mathbb{R}^2 .

Excluding the momenta p_t and p_x from (3.5) we find explicit formulae for particle trajectories

$$x(t) = M/2K, \quad \text{for } P = 0 \quad (3.9)$$

and

$$x(t) = \left[K - \sqrt{\kappa^2 + (rP)^2 \exp(-2t/r)} \right] / P, \quad \text{for } P \neq 0, \quad (3.10)$$

where (3.10) takes into account that $K - xP > 0$.

The space of trajectories defined by (3.9) and (3.10) represents the phase-space Γ_p .

B. Choice of observables

To satisfy all required criteria for observables, we parametrize Γ_p by the coordinates $(q, p) \in \mathbb{R}^2$ as follows

$$P = p, \quad K = pq - \kappa, \quad M = pq^2 - 2\kappa q. \quad (3.11)$$

The integrals (3.11) satisfy the algebra (3.6), if $\{p, q\} = 1$.

To compare quantum dynamics of V_p and V_h systems, let us bring their observables to the same functional form. It can be achieved in two steps:

First, we change parametrization of the phase-space Γ_p as follows

$$q =: \cot \frac{\sigma}{2}, \quad p =: (1 - \cos \sigma)(I + \kappa \cot \frac{\sigma}{2}), \quad (3.12)$$

where $0 < \sigma < 2\pi$ and $I \in \mathbb{R}$.

Second, we rewrite the observables (3.11) in terms of new canonical variables (I, σ) and redefine them. The final result is

$$I_0 := \frac{1}{2}(M + P) = I, \quad (3.13)$$

$$I_1 := \frac{1}{2}(M - P) = I \cos \sigma - \kappa \sin \sigma, \quad (3.14)$$

$$I_2 := K = I \sin \sigma + \kappa \cos \sigma. \quad (3.15)$$

Since $\{\sigma, I\} = 1$, the commutation relations for I_a resulting from (3.6) are identical to the commutation relations (2.13) for J_a ($a = 0, 1, 2$).

Comparing (3.13 - 3.15) with (2.17) we can see that I_a and J_a ($a = 0, 1, 2$) have the same functional forms, but they are different because the range of parameter β is $0 \leq \beta < 2\pi$, whereas the range of σ reads $0 < \sigma < 2\pi$. This difference results from the difference between the topologies of phase-spaces of V_h and V_p systems: Γ_h is the hyperboloid (2.15), whereas Γ_p is the hyperboloid (3.8) without one line. Therefore, the phase-space Γ_p cannot be invariant under the action of $SO_0(1, 2)$ group. This may be already seen in the context of space-times,

since V_p is only a subspace of V_h due to the isometric immersion map (2.3). In fact, the Killing vector field generated by the transformation (3.4) is not complete on V_p , whereas the vector fields generated by (3.2) and (3.3) are well defined globally (see, App. B). Therefore, the dynamical integral M is not well defined globally. Let us make the assumption that each classical observable should be a globally well defined function on a physical phase-space. Then, the set of observables of V_p system consists of only the integrals P and K satisfying the algebra (see, (3.6))

$$\{P, K\} = P. \quad (3.16)$$

Eq. (3.16) defines a solvable subalgebra of $sl(2, \mathbb{R})$ algebra.

The algebra (3.16) is isomorphic to the algebra $\mathfrak{aff}(1, \mathbb{R})$ of the affine group $\mathfrak{Aff}(1, \mathbb{R})$. If we denote the span of the algebra (3.16) by $\langle P, K \rangle$ and the span of $\mathfrak{aff}(1, \mathbb{R})$ by $\langle A, B \rangle$, the algebra isomorphism is defined by $A := -K$ and $B := -P$. The algebra $\mathfrak{aff}(1, \mathbb{R})$ is defined by the commutation relation

$$\{A, B\} = B. \quad (3.17)$$

One can easily show that the center of $\mathfrak{Aff}(1, \mathbb{R})$ is an identity element of this group. Thus, $\mathfrak{Aff}(1, \mathbb{R})$ is the only Lie group with $\mathfrak{aff}(1, \mathbb{R})$ as its Lie algebra. This circumstance makes unique the choice of the symmetry group of the phase-space Γ_p .

The algebra $\mathfrak{aff}(1, \mathbb{R})$ is quite different from the local symmetry (3.6) of Γ_p . The relationship between local and global symmetries which occurs in V_h case does not exist in the present case.

C. Quantum dynamics on plane

In gravitational systems the global and local symmetries may easily happen to be incompatible. An example is our V_p system of a free particle in space-time with removable type singularities. In such cases our GTQ method needs modification to be applicable. We propose to complete the set of conditions defining the algebra of observables, Sec. IIB, by the following one: *(iv) algebra of observables is consistent with the global symmetry of phase-space.*

To quantize the algebra (3.17) we use the fact that all unitary irreducible representations of $\mathfrak{Aff}(1, \mathbb{R})$ group are known. They were discovered already in 1947 by Gel'fand and Najmark [22]. In what follows we adopt the Vilenkin version [17]. There exist only two (nontrivial) unitarily non-equivalent representations

$$U_s(g) : \mathcal{H}_\Lambda \longrightarrow \mathcal{H}_\Lambda, \quad s = -, + \quad (3.18)$$

where $g \in \mathfrak{Aff}(1, \mathbb{R})$ and where \mathcal{H}_Λ is the Hilbert space defined as follows: We introduce the space Λ

$$\Lambda := C_0^\infty(\mathbb{R}_+), \quad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}, \quad (3.19)$$

with the scalar product given by

$$\langle \varphi_1 | \varphi_2 \rangle := \int_0^\infty \overline{\varphi_1(x)} \varphi_2(x) \frac{dx}{x}, \quad \varphi_1, \varphi_2 \in \Lambda. \quad (3.20)$$

\mathcal{H}_Λ is the Hilbert space obtained by completion of Λ with respect to the scalar product (3.20). The operators $U_s(g)$ are defined as follows [17]

$$U_s[g(a, b)]\psi(x) := \exp(-isbx) \psi(ax), \quad \psi \in \mathcal{H}_\Lambda, \quad s = -, + \quad (3.21)$$

where $g(a, b) \in \mathfrak{Aff}(1, \mathbb{R})$ and $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$, parametrize the group elements.

It is easy to check that (3.21) is a representation of $\mathfrak{Aff}(1, \mathbb{R})$ group.

Since the measure $x^{-1}dx$ in (3.21) is invariant with respect to $x \rightarrow ax$, we obtain

$$\langle U_s[g(a, b)]\psi_1 | U_s[g(a, b)]\psi_2 \rangle = \int_0^\infty \overline{\psi_1(ax)} \psi_2(ax) \frac{dx}{x} = \int_0^\infty \overline{\psi_1(x)} \psi_2(x) \frac{dx}{x} = \langle \psi_1 | \psi_2 \rangle \quad (3.22)$$

for all $\psi_1, \psi_2 \in \mathcal{H}_\Lambda$, which shows that (3.21) defines a unitary representation. Making use of the reasoning of Ref. [17] one can prove that the unitary representation (3.21) is irreducible.

The application of Stone's theorem (strong form) to (3.21) defines two sets of operators \hat{A}_s and \hat{B}_s ($s = -, +$).

$$\frac{d}{dt}\bigg|_{t=0} U_s[g(a(t), 0)]\varphi(x) = x \frac{da(0)}{dt} \frac{d\varphi(x)}{dx} = x \frac{d}{dx} \varphi(x) = i(-ix \frac{d}{dx} \varphi(x)) =: i\hat{A}_s \varphi(x) \quad (3.23)$$

and

$$\frac{d}{dt}\bigg|_{t=0} U_s[g(1, b(t))]\varphi(x) = -isx \frac{db(0)}{dt} e^{-isb(0)x} \varphi(x) = i(-sx) \varphi(x) =: i\hat{B}_s \varphi(x), \quad (3.24)$$

where $t \rightarrow a(t)$ and $t \rightarrow b(t)$ with the boundary conditions $a(0) = 1$, $da(0)/dt = 1$ and $b(0) = 0$, $db(0)/dt = 1$, respectively, are two integral curves on $\mathfrak{Aff}(1, \mathbb{R})$.

Equations (3.23) and (3.24) define the domains for the operators \hat{A}_s and \hat{B}_s . One can prove (see, App. C) that these operators are essentially self-adjoint on the space Λ defined by (3.19). One can also verify that

$$[\hat{A}_s, \hat{B}_s]\varphi = -i\hat{B}_s\varphi, \quad \varphi \in \Lambda, \quad s = -, + \quad (3.25)$$

which demonstrates that (3.25) is the representation of the algebra $\mathfrak{aff}(1, \mathbb{R})$ defined by (3.17).

Therefore, there are possible only two (up to unitary equivalence) quantum dynamics corresponding to a single classical dynamics of V_p system. We can use either the representation $U_+(g)$ or $U_-(g)$. (We ignore the trivial one-dimensional representation mentioned in [17].)

The quantization of V_p system is now complete.

To appreciate the quantization requirement that representation of the algebra of observables should be integrable to the unitary representation of the symmetry group of the system, let us consider again the representation of $sl(2, \mathbb{R})$ algebra satisfied by I_a ($a = 0, 1, 2$) observables (3.13 - 3.15). Since I_a and J_a ($a = 0, 1, 2$) have the same functional forms and have almost everywhere the same ranges, the representation of I_a observables is defined by (2.18 - 2.22) with \hat{J}_a replaced by \hat{I}_a and β replaced by σ . However, there exists no value of the parameter θ in (2.22) which can lead to the algebra representation integrable to the unitary representation of the symmetry group $\mathfrak{Aff}(1, \mathbb{R})$. It is so because $sl(2, \mathbb{R})$ is not the algebra of $Aff(1, \mathbb{R})$.

IV. DYNAMICS OF MASSLESS PARTICLE

A. Massless particle on hyperboloid

To obtain the description of dynamics of a massless particle on hyperboloid we examine taking the limit $\kappa \rightarrow 0$, i.e. $m \rightarrow 0$, in Sec. II. The inspection of classical and quantum dynamics of V_h system reveals that apart from Eq. (2.5) for the Lagrangian, all equations can be considered in the limit $m \rightarrow 0$:

The phase-space Γ_h defined by (2.15) turns into two cones

$$J_1^2 + J_2^2 - J_0^2 = 0. \quad (4.1)$$

with a common vertex V defined by $J_0 = 0 = J_1 = J_2$.

It is clear that each cone is invariant under the action of $SO_0(1, 2)$ group.

Each point of (4.1) different from V labels uniquely the trajectory of a particle on hyperboloid (2.4). The set of trajectories (straight lines) is the set of generatrices of the hyperboloid (2.4).

Parametrizing (4.1) by J_a in the form (2.17) with $\kappa = 0$ leads to (2.18 - 2.20) with $\kappa = 0$ as well.

The quantum Casimir operator (2.26) now reads

$$\hat{C}\psi = \frac{1}{4}\psi, \quad \psi \in \Omega. \quad (4.2)$$

There is no problem with taking $\kappa \rightarrow 0$ in (2.27 - 2.29) too. The only problem is the form of the Lagrangian (2.5) because m occurs as a factor. We can avoid this difficulty by choosing the Lagrangian which does not depend explicitly on the mass of a particle [4, 24]

$$\mathcal{A} = \int_{\tau_1}^{\tau_2} L(\tau) d\tau, \quad L(\tau) := -\frac{1}{2\lambda(\tau)} g_{\mu\nu}(x^0(\tau), x^1(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau), \quad (4.3)$$

where τ is an evolution parameter, $\dot{x}^\mu = dx^\mu/d\tau$ and λ plays the role of Lagrangian multiplier. The action (4.3) is invariant under reparametrization $\tau \rightarrow f(\tau)$, $\lambda(\tau) \rightarrow \lambda(\tau)/\dot{f}(\tau)$. This gauge symmetry leads to dynamics constrained by (2.14) with $m = 0$ and consequently to (2.15) with $\kappa = 0$, i.e. to Eq. (4.1). Thus the dynamics of a massless particle defined by (2.5) and (4.3) are equivalent. It appears that the massless particle dynamics of quantum V_p system may be described by the continuous (integral case) Bargmann's C_q^0 class with $q = 1/4$ (see, the last paragraph of Sec. IIC).

Thus, the principal series irreducible unitary representation of $SO_0(1, 2)$ group appears to be able to describe quantum dynamics of both massive and massless particle on hyperboloid.

However, we have not analysed the vertex of (4.1) in the context of particle dynamics carefully enough. The subtlety is that in case of a massless particle the vertex cannot be used to specify the trajectory of a particle by making use of (2.16). For $p_\rho \neq 0$ the Eq. (2.16) has no solutions and for $p_\rho = 0$ it has infinitely many. The removal of the vertex from (4.1) turns the phase-space of V_p system into two separate cones \mathcal{C}_+ and \mathcal{C}_- with $J_0 > 0$ and $J_0 < 0$, respectively. This procedure splits the system into two parts and each part can be quantized independently. The corresponding quantum systems have been already found [4]: the quantum system connected with \mathcal{C}_+ may be described by the discrete series D_+ (having positive spectrum of \hat{J}_0) of the irreducible unitary representation of $SO_0(1, 2)$ group, whereas

the discrete series D_- (with negative spectrum of \hat{J}_0) may be used to represent the quantum system connected with \mathcal{C}_- . Thus the whole quantum system may be described by $D_- \oplus D_+$ representation (see, Sec. 2 of [4] for more details).

The role of the vertex V is of primary importance. If we take it to belong to the phase-space, the corresponding quantum system will be unique and may be described by an irreducible representation of $SO_0(1, 2)$ group. The phase-space without the vertex leads to infinitely many reducible representations of $SO_0(1, 2)$. Therefore, taking $\kappa \rightarrow 0$ at the classical level leads to the latter. Taking the limit not at the classical, but at the quantum level gives the former.

B. Massless particle on plane

In case of dynamics on plane (see, Sec. III) an action integral is defined by (4.3) and the one-sheet hyperboloid (3.8), in the limit $m \rightarrow 0$, turns into ‘one-sheet cone’

$$K^2 - PM = 0. \quad (4.4)$$

Since the dynamics requires $K > 0$ for $P = 0$, we have to remove the line ($P = 0 = K$) from (4.4) to get the physical phase-space Γ_p . As the result, the phase-space splits into two disconnected parts \mathcal{P}_+ and \mathcal{P}_- with $P > 0$ and $P < 0$, respectively. The observables P and K are well defined globally on \mathcal{P}_+ and \mathcal{P}_- , and the corresponding A and B observables satisfy $\mathfrak{aff}(1, \mathbb{R})$ algebra (3.17) with $\mathfrak{Aff}(1, \mathbb{R})$ as the symmetry group.

Since at the quantum level there is no explicit dependence on the parameter κ , taking $m \rightarrow 0$ is trivial.

Therefore, the quantum dynamics on plane of the massless particle may be described by the representation $T_- \oplus T_+$, where T_- and T_+ denote the representations corresponding to $s = -$ and $s = +$, respectively.

In the present case there is no ambiguity connected with the vertex of (4.4) since the removed line ($P=0=K$) includes the vertex.

V. CONCLUSIONS

Test particle is a tool which may be used in the theory of classical gravity to examine causal geodesics of singular space-times. We have tried to quantize the particle dynamics to see what can one do to avoid problems connected with *removable* singularities of space-time in quantum theory. Our main result is that taking account of *global* properties of space-time makes possible the imposition of quantum rules into the dynamics of a particle. It is clear that we have not tried to quantize gravitational field, but only the dynamics of a particle.

Local properties of a given space-time described by metric tensor and Lie algebra of the Killing vector fields do not specify the system uniquely because space-times with different global properties may have isometric Lie algebras [20, 21]. Presented results show that the topology of space-time carries the information not only on the symmetry group but also indicates which local properties of the system should be used in the quantization procedure. Our results are consistent with the fact that quantum theory is a global theory in its nature. We suggest that its consolidation with classical gravity should take into account both local and global properties of space-time. The Einstein equations being partial differential equations cannot specify the space-time topology, but only its local properties. Fortunately, the

mathematics of low dimensional manifolds offers a full variety of topologies for space-time models consistent with local properties of a given space-time [25, 26].

Generalization of our results to the four-dimensional de Sitter space-times seems to be straightforward. The space-time with topology $\mathbb{R}^1 \times \mathbb{R}^3$, the four dimensional analog of V_p , is geodesically incomplete and it can be embedded isometrically [8] into the space-time with topology $\mathbb{R}^1 \times \mathbb{S}^3$, corresponding to V_h , by generalization of the mapping (2.3). The quantum dynamics of a particle on four dimensional hyperboloid in five dimensional Minkowski space is presented in [27]. It seems that quantization of dynamics of a particle on de Sitter space-time with topology $\mathbb{R}^1 \times \mathbb{R}^3$ may be carried out by analogy to the quantization of V_p system. First of all one should find the set of all Killing vector fields which are complete. They would help to identify the algebra of globally well defined dynamical integrals and the symmetry group of phase-space. Unitary representations of the symmetry group may be used to define quantum dynamics of a particle. It is clear that examination of dynamics of a particle in four dimensional space-time is much more complicated than in two dimensional case, but it should be feasible if the number of globally well defined dynamical integrals is high enough. We expect that application of our method to the four dimensional case should lead to the conclusion similar in its essence to the two dimensional case.

Our paper concerns removable type singularities of space-time. Great challenge is an extension of our analysis to space-times with essential type singularities, i.e. including not only incomplete geodesics, but also blowing up Riemann tensor components or curvature invariants [28]. The FLRW type universes appear to be good candidates to begin with, since their properties are well known [29]. Our method of analysing particle dynamics by making use of embeddings into the Minkowski space extends to higher dimensions. There exist theorems of differential geometry [30, 31, 32, 33] that every curved four-dimensional space-time can be embedded isometrically into a flat pseudo-Euclidean space E_N with $5 \leq N \leq 10$.

Recently, Heller and Sasin put forward the idea of modeling space-time by the Connes noncommutative geometry. With this new idea one can try to coup with space-time singularities and try to establish the relationship with quantum description [34, 35].

Completely different approach has been developed by Ashtekar and his collaborators (see, [36] and references therein). This non-perturbative and background-independent theory of quantum gravity seems to be free of problems connected with space-time singularities [37, 38], and it seems to reproduce (in its classical limit) the Einstein theory of gravity.

Finally, let us make some general comments concerning the ambiguity problems of the canonical quantization procedure. It is known that quantization of a system with constraints is a highly nonunique procedure. One may quantize first and then impose the constraints, or vice versa. Having fixed the phase-space one may quantize geometrically, group theoretically, by making use of coherent states or by some mixture of all these methods. Only quantization of the simplest mechanical systems leads to similar or identical results. The ambiguity may be reduced by making use of global properties of classical system. In principle, the ambiguity should be removed by comparison of predictions with experimental data. In case of the systems with a few degrees of freedom and with non-trivial topology of phase-space the discussion of the ambiguity problem was recently done in Refs. [39, 40]. An extension of this discussion will be published elsewhere [16].

APPENDIX A: REPRESENTATION ALGEBRA ON HYPERBOLOID

Let $L^2(\mathbb{S})$ denotes the Hilbert space of square integrable complex functions on a unit circle with the inner product

$$\langle \varphi | \psi \rangle = \int_0^{2\pi} d\beta \overline{\varphi(\beta)} \psi(\beta), \quad \varphi, \psi \in L^2(\mathbb{S}). \quad (\text{A1})$$

In what follows we outline the prove that representation of $sl(2, R)$ algebra defined by

$$\hat{J}_0 \psi(\beta) := \frac{1}{i} \frac{d}{d\beta} \psi(\beta), \quad \beta \in \mathbb{S}, \quad \psi \in \Omega_\theta \quad \theta \in \mathbb{R}, \quad (\text{A2})$$

$$\hat{J}_1 \psi(\beta) := \left[\cos \beta \hat{J}_0 - \left(\kappa - \frac{i}{2} \right) \sin \beta \right] \psi(\beta), \quad (\text{A3})$$

$$\hat{J}_2 \psi(\beta) := \left[\sin \beta \hat{J}_0 + \left(\kappa - \frac{i}{2} \right) \cos \beta \right] \psi(\beta), \quad (\text{A4})$$

where

$$\Omega_\theta := \{ \psi \in L^2(\mathbb{S}) \mid \psi \in C^\infty[0, 2\pi], \psi^{(n)}(0) = e^{i\theta} \psi^{(n)}(2\pi), n = 0, 1, 2, \dots \}, \quad (\text{A5})$$

is essentially self-adjoint.

It is clear that Ω_θ is a dense invariant common domain for \hat{J}_a ($a = 0, 1, 2$). Since the functional form of \hat{J}_a does not depend on θ and since $\exp(-i\theta) \cdot \exp(i\theta) = 1$, the operators are symmetric on Ω_θ :

An elementary proof includes integration by parts of one side of

$$\langle \phi_1 | \hat{J}_a \phi_2 \rangle = \langle \hat{J}_a \phi_1 | \phi_2 \rangle, \quad \phi_1, \phi_2 \in \Omega_\theta \quad (\text{A6})$$

followed by making use of the property

$$\phi(0) = \exp(i\theta) \phi(2\pi), \quad \phi \in \Omega_\theta. \quad (\text{A7})$$

The domains $D(\hat{J}_a^*)$ of the adjoint \hat{J}_a^* of \hat{J}_a consists of functions ψ_a which satisfy the condition

$$\psi_a(0) = \exp(i\theta) \psi_a(2\pi), \quad \psi_a \in D(\hat{J}_a^*) \subset L^2(\mathbb{S}) \quad (\text{A8})$$

for $a = 0, 1, 2$.

The main idea of the proof [12] is to show that the only solutions to the equations

$$\hat{J}_a^* f_{a\pm} = \pm i f_{a\pm}, \quad f_{a\pm} \in D(\hat{J}_a^*), \quad a = 0, 1, 2 \quad (\text{A9})$$

are $f_{a\pm}(\beta) = 0$, i.e. the deficiency indices of \hat{J}_a on Ω_θ satisfy $n_{a+} = 0 = n_{a-}$ (for $a = 0, 1, 2$).

The equation (A9) for $a = 0$ reads

$$\frac{1}{i} \frac{d}{d\beta} f_{0\pm}(\beta) = \pm i f_{0\pm}(\beta) \quad (\text{A10})$$

and its general normalized solution is

$$f_{0\pm}(\beta) = C_{0\pm} \exp(\mp\beta), \quad C_{0+} := \sqrt{2/(1 - \exp(-4\pi))}, \quad C_{0-} := \sqrt{2/(\exp(4\pi) - 1)}. \quad (\text{A11})$$

The solutions (A11) does not satisfy (A8). Thus the only solution to (A10) is $f_{0\pm}(\beta) = 0$.

For $a = 1$ the equation (A9) can be written as

$$(\cos \beta \frac{d}{d\beta} - r \sin \beta + \lambda_{\pm}) f_{1\pm}(\beta) = 0, \quad (\text{A12})$$

where $r = 1/2 + \kappa i$, $\kappa \in \mathbb{R}$, $\lambda_{\pm} = 1$ or -1 for f_{1+} or f_{1-} , respectively. One can verify that the general solution of (A12) reads

$$f_{1\pm}(\beta) = C_{1\pm} |\cos \beta|^{-r} |\tan(\frac{\beta}{2} + \frac{\pi}{4})|^{-\lambda_{\pm}}, \quad (\text{A13})$$

where $C_{1\pm}$ are complex constants.

The immediate calculations show that for $C_{1\pm} \neq 0$

$$\lim \Re f_{1+}(\beta) = \infty = \lim \Im f_{1+}(\beta) \quad \text{as} \quad \beta \rightarrow \frac{3}{2}\pi \pm \quad (\text{A14})$$

and

$$\lim \Re f_{1-}(\beta) = \infty = \lim \Im f_{1-}(\beta) \quad \text{as} \quad \beta \rightarrow \frac{\pi}{2} \pm. \quad (\text{A15})$$

Therefore $f_{1\pm}$ are not square integrable and the only solutions of (A12) are $f_{1\pm} = 0$.

The equation (A9) for $a = 2$ has the form

$$(\sin \beta \frac{d}{d\beta} + r \cos \beta + \lambda_{\pm}) f_{2\pm}(\beta) = 0, \quad (\text{A16})$$

where $r = 1/2 + \kappa i$ and $\lambda_{\pm} = 1$ or -1 , for f_{2+} or f_{2-} , respectively.

The general solution to (A16) is

$$f_{2\pm}(\beta) = C_{2\pm} |\sin \beta|^{-r} |\tan \frac{\beta}{2}|^{-\lambda_{\pm}}, \quad (\text{A17})$$

where $C_{2\pm}$ are complex constants.

The standard calculations yield

$$\lim \Re f_{2+}(\beta) = \infty = \lim \Im f_{2+}(\beta) \quad \text{as} \quad \beta \rightarrow 0 + \quad \text{or} \quad \beta \rightarrow 2\pi - \quad (\text{A18})$$

and

$$\lim \Re f_{2-}(\beta) = \infty = \lim \Im f_{2-}(\beta) \quad \text{as} \quad \beta \rightarrow \pi \pm. \quad (\text{A19})$$

Thus, $f_{2\pm}$ are not square integrable unless $C_{2\pm} = 0$.

This finishes the proof, the detailed verification of consecutive steps being left to the reader.

APPENDIX B: GLOBAL TRANSFORMATIONS ON PLANE

The transformations (3.2), (3.3) and (3.4) of Sec. IIIA lead, respectively, to the following infinitesimal generators

$$X_1 = \partial/\partial x, \quad (\text{B1})$$

$$X_2 = -r\partial/\partial t + x\partial/\partial x, \quad (\text{B2})$$

$$X_3 = -2rx\partial/\partial t + (x^2 + r^2 \exp(-2t/r))\partial/\partial x. \quad (\text{B3})$$

The one-parameter group generated by X_3 is defined by the solution of the Lie equations

$$\frac{dt}{db_3} = -2rx, \quad (\text{B4})$$

$$\frac{dx}{db_3} = x^2 + r^2 \exp(-2t/r), \quad (\text{B5})$$

$$t|_{b_1=0=b_2=b_3} = t_0 \quad (\text{B6})$$

$$x|_{b_1=0=b_2=b_3} = x_0. \quad (\text{B7})$$

(In what follows we use $\epsilon := b_3$ to simplify notation.)

Acting of $\partial/\partial\epsilon$ on (B5) and making use of (B4) gives

$$\frac{d^2x}{d\epsilon^2} - 6x\frac{dx}{d\epsilon} + 4x^3 = 0. \quad (\text{B8})$$

To reduce the order of (B8) we introduce $p := dx/d\epsilon$, which leads to the equation

$$p\frac{dp}{dx} - 6xp + 4x^3 = 0. \quad (\text{B9})$$

Eq. (B9) becomes homogeneous for $z^2 := p$, since we get

$$\frac{dz}{dx} = \frac{3xz^2 - 2x^3}{z^3}. \quad (\text{B10})$$

Substitution $z := ux$ into (B10) gives

$$\frac{u^3 du}{-u^4 + 3u^2 - 2} = \frac{dx}{x}. \quad (\text{B11})$$

One more substitution $v := u^2$ turns (B11) into

$$\left(\frac{1}{v-1} - \frac{2}{v-2}\right)dv = \frac{2}{x}dx. \quad (\text{B12})$$

Solution to (B12) reads

$$\frac{v-1}{(v-1)^2} = Cx^2, \quad (\text{B13})$$

where $R^1 \ni C > 0$ is a constant.

Making use of $p = dx/d\epsilon$, $p = z^2$, $z = ux$ and $v = u^2$ turns (B13) into an algebraic equation

$$\left(\frac{dx}{d\epsilon}\right)^2 - (4x^2 + D)\frac{dx}{d\epsilon} + 4x^4 + Dx^2 = 0, \quad (\text{B14})$$

where $D := 1/C$.

Eq. (B14) splits into two first-order real equations. One of them has the form (Analysis of the other one can be done by analogy.)

$$2\frac{dx}{d\epsilon} = 4x^2 + D - \sqrt{D(4x^2 + D)}. \quad (\text{B15})$$

The solution to (B15) reads

$$\epsilon(x) = 2 \int \frac{dx}{4x^2 + D - \sqrt{D(4x^2 + D)}} = \frac{1}{A - x - \sqrt{x^2 + A^2}} + B, \quad (\text{B16})$$

where $A = \sqrt{D}/2$ and B are real constants.

Eq. (B16) leads to

$$x(\epsilon) = \frac{A(\epsilon - B)[A(\epsilon - B) - 1] + 1}{2(\epsilon - B)[A(\epsilon - B) - 1]}. \quad (\text{B17})$$

Eq. (B17) represents one of the solutions of (B5). It is not defined for $\epsilon = B$ because

$$\lim_{\epsilon \rightarrow B-} x(\epsilon) = +\infty, \quad \lim_{\epsilon \rightarrow B+} x(\epsilon) = -\infty. \quad (\text{B18})$$

Since (B17) is not defined for all $\epsilon \in \mathbb{R}$, we conclude that the vector field X_3 is not complete on the plane.

One can easily solve the Lie equations corresponding to (B1) and (B2). The solutions, respectively, read

$$(t, x) \longrightarrow (t, x + b_0) \quad (\text{B19})$$

and

$$(t, x) \longrightarrow (t - rb_1, x \exp b_1). \quad (\text{B20})$$

Both (B19) and (B20) describe one-parameter global transformations on V_p well defined for any $b_0, b_1 \in \mathbb{R}$. Therefore, the vector fields X_1 and X_2 are complete on the plane.

APPENDIX C: REPRESENTATION ALGEBRA ON PLANE

In what follows we consider only the case $\hat{A} \equiv \hat{A}_-$ and $\hat{B} \equiv \hat{B}_-$ (another case can be done by analogy).

We give the proof that representation of the algebra

$$\{A, B\} = B \quad (\text{C1})$$

defined by

$$\hat{B}\phi(x) := x\phi(x), \quad \hat{A}\phi(x) := -ix\frac{d}{dx}\phi(x) \quad x \in \mathbb{R}_+, \quad \phi \in \Lambda = C_0^\infty(\mathbb{R}_+) \subset \mathcal{H}_\Lambda \quad (\text{C2})$$

with

$$\langle \phi_1 | \phi_2 \rangle = \int_0^\infty \overline{\phi_1(x)} \phi_2(x) \frac{dx}{x}, \quad \phi_1, \phi_2 \in \Lambda \quad (\text{C3})$$

is essentially self-adjoint on Λ (the space \mathcal{H}_Λ denotes the completion of Λ with respect to the inner product (C3)):

It is easy to see that the representation (C2) and is symmetric on a common invariant dense domain Λ .

To examine the self-adjointness of \hat{A} we solve the equation

$$\hat{A}^* f_\pm(x) = \pm i f_\pm(x), \quad f_\pm \in D(\hat{A}^*) \subset \mathcal{H}_\Lambda \quad (\text{C4})$$

to find the deficiency indices $n_+(\hat{A})$ and $n_-(\hat{A})$. The solution to (C4) reads

$$f_\pm(x) = a_\pm x^{\pm 1}, \quad (\text{C5})$$

where $a_\pm \in \mathbb{C}$.

It is clear that f_\pm are not in \mathcal{H}_Λ unless $a_\pm = 0$. Thus $n_+(\hat{A}) = 0 = n_-(\hat{A})$, which means [12] that \hat{A} is essentially self-adjoint on Λ .

The case of \hat{B} operator is trivial since

$$\hat{B}^* g_\pm(x) = \pm i g_\pm(x), \quad g_\pm \in D(\hat{B}^*) \subset \mathcal{H}_\Lambda \quad (\text{C6})$$

reads $(x \mp i)g_\pm(x) = 0$. Its only solutions are $g_\pm(0) = 0$, which proves that $n_+(\hat{B}) = 0 = n_-(\hat{B})$. Therefore, Eqs. (C2) and (C3) define an essentially self-adjoint representation of (C1) algebra.

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- [1] Jorjadze G and Piechocki W 1998 *Class. Quant. Grav.* **15** L41
 - [2] Jorjadze G and Piechocki W 1999 *Theor. Math. Phys.* **118** 183
 - [3] Jorjadze G and Piechocki W 1999 *Phys. Lett.* **B461** 183
 - [4] Jorjadze G and Piechocki W 1999 *Phys. Lett.* **B448** 203
 - [5] Piechocki W 2002 *Phys. Lett* **B526** 127
 - [6] Isham C J 1984 ‘Topological and global aspects of quantum theory’ in: *Relativity, Groups and Topology II*, Eds. B. S. DeWitt and R. Stora (Amsterdam: North-Holland)
 - [7] Kirillov A A 1976 *Elements of the Theory of Representations* (Berlin: Springer)
 - [8] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press)
 - [9] Stephani H 1982 *General Relativity* (Cambridge: Cambridge University Press)
 - [10] Souriau J-M 1997 *Structure of Dynamical Systems* (Boston: Birkhäuser)
 - [11] Stone M H 1932 *Annals Math.* **33** 643

- [12] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* (New York: Academic Press)
- [13] Bargmann V 1947 *Annals Math.* **48** 568
- [14] Loll R 1990 *Phys. Rev. D* **41** 3785
- [15] Plyushchay M 1993 *J. Math. Phys.* **34** 3954
- [16] Gazeau J-P and Piechocki W, in preparation
- [17] Vilenkin N Ja and Klimyk A U 1991 *Representation of Lie Groups and Special Functions* (Dodrecht: Kluwer Academic Publishers) Vol.1
- [18] Sally P J 1967 *Memoirs Amer. Math. Soc.* **69**
- [19] Wong M K F 1974 *J. Math. Phys.* **15** 25
- [20] Lachièze-Rey M and Luminet J-P 1995 *Phys. Rep.* **254** 135
- [21] Pontrjagin L 1946 *Topological Groups* (Princeton: Princeton University Press)
- [22] Gel'fand I M and Najmark M A 1947 *Dokl. Akad. Nauk SSSR* **55** 567
- [23] Wigner E 1939 *Annals Math.* **40** (1939) 149
- [24] Brink L Di Vecchia P and Hove P 1977 *Nucl. Phys.* **B118** 76
- [25] Thurston W P and Week J R 1984 *Sci. Am.* **251** 108 (July issue)
- [26] Thurston W P 1997 *Three-Dimensional Geometry and Topology* (Princeton: Princeton University Press)
- [27] Jorjadze G and Piechocki W 2000 *Phys. Lett.* **B476** 134
- [28] Senovilla J M M 1998 *Gen. Relativ. Gravit.* **30** 701
- [29] Ellis G F R and MacCallum M A H 1969 *Comm. Math. Phys.* **12** 108
- [30] Eisenhart L P 1948 *Riemannian Geometry* (Princeton: Princeton University Press)
- [31] Plebański J 1967 'On Conformally Equivalent Riemannian Spaces' Report of the CIEAIPN, México
- [32] Seahra S S and Wesson P S 2003 *Preprint* gr-qc/0302015
- [33] Lachièze-Rey M 2000 *Astron. Astrophys.* **364** 894
- [34] Heller M and Sasin S 1999 *General Relat. Grav.* **31** 555
- [35] Heller M and Sasin W 1996 *J. Math. Phys.* **37** 5665
- [36] Ashtekar A 2002 *Preprint* math-ph/0202008
- [37] Bojowald M 2002 *Preprint* gr-qc/0202077
- [38] Bojowald M 2003 *Preprint* gr-qc/0303073
- [39] Bojowald M, Kastrup H A, Schramm F and Strobl T 2000 *Phys. Rev. D* **62** 44026
- [40] Bojowald M Strobl T 2000 *J. Math. Phys.* **41** 2537